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# PERIODS OF AUTOMORPHIC FORMS: THE CASE OF $(\mathrm{GL}_{n+1}\times\mathrm{GL}_n, \mathrm{GL}_n)$

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This note is a report on a joint work with Shunsuke Yamana [2]. Details will appear elsewhere.

Let $G$ be a connected reductive algebraic group over a number field $F$ and $G'$ a closed subgroup of $G$ over $F$. Let $\mathscr{A}(G)$ and $\mathscr{A}(G')$ denote the spaces of automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. We will consider the period integral

$$\mathbf{P}^{G'}(\varphi\otimes\varphi') := \int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g)\,\mathrm{d}g$$

for $\varphi\in\mathscr{A}(G)$ and $\varphi'\in\mathscr{A}(G')$. Let $\pi\subset\mathscr{A}(G)$ and $\pi'\subset\mathscr{A}(G')$ be irreducible subrepresentations. If $\mathbf{P}^{G'}(\varphi\otimes\varphi')$ converges for all $\varphi\in\pi$ and $\varphi'\in\pi'$, then

$$\mathbf{P}^{G'}|_{\pi\otimes\pi'}\in\mathrm{Hom}_{\Delta G'(\mathbb{A})}(\pi\otimes\pi',\mathbb{C}).$$

We say that $\pi\otimes\pi'$ is $\Delta G'$-distinguished (with respect to $\mathbf{P}^{G'}$) if $\mathbf{P}^{G'}|_{\pi\otimes\pi'}\neq 0$.

In this note, we consider the case $G=\mathrm{GL}_{n+1}$ and $G'=\mathrm{GL}_n$, which was studied by Jacquet, Piatetski-Shapiro and Shalika.

**Theorem 1** (Jacquet-Piatetski-Shapiro-Shalika). *If $\varphi\in\mathscr{A}^{\mathrm{cusp}}(G)$ and $\varphi'\in\mathscr{A}^{\mathrm{cusp}}(G')$, then*

$$\mathbf{P}^{G'}(\varphi\otimes\varphi'_s)=I(s,\varphi,\varphi') := \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})} W^{\psi}(g,\varphi)W^{\bar\psi}(g,\varphi')|\det g|^s\,\mathrm{d}g.$$

*Here, $\mathscr{A}^{\mathrm{cusp}}(G)$ and $\mathscr{A}^{\mathrm{cusp}}(G')$ denote the spaces of cusp forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively, $\varphi'_s=\varphi'\cdot|\det|^s$ for $s\in\mathbb{C}$, $N\subset G$ and $N'\subset G'$ are upper triangular unipotent subgroups, $W^{\psi}(g,\varphi)$ is a Whittaker function (with respect to a nontrivial character $\psi$ of $F\backslash\mathbb{A}$) defined by*

$$W^{\psi}(g,\varphi)=\int_{N(F)\backslash N(\mathbb{A})}\varphi(ug)\overline{\psi(u_{1,2}+u_{2,3}+\cdots+u_{n,n+1})}\,\mathrm{d}u$$

*and $W^{\bar\psi}(g,\varphi')$ is defined similarly. The left-hand side converges for all $s$ and the right-hand side converges for $\Re s\gg 0$. Moreover, if $\varphi=$*

$\otimes_v \varphi_v \in \pi \subset \mathscr{A}^{\text{cusp}}(G)$ *and* $\varphi' = \otimes_v \varphi'_v \in \pi' \subset \mathscr{A}^{\text{cusp}}(G')$, *then*

$$I(s,\varphi,\varphi') = L\left(s+\tfrac{1}{2},\pi\times\pi'\right)\prod_v \frac{I(s,W_{\varphi_v}^{\psi_v},W_{\varphi'_v}^{\bar{\psi}_v})}{L\left(s+\tfrac{1}{2},\pi_v\times\pi'_v\right)}.$$

*In particular,* $\pi\otimes\pi'$ *is* $\Delta G'$*-distinguished if and only if*

$$L\left(\tfrac{1}{2},\pi\times\pi'\right)\neq 0.$$

The last assertion is a special case of the Gan-Gross-Prasad conjecture [1]. We also remark that $I(s,\varphi,\varphi')$ makes sense for any automorphic forms $\varphi$ and $\varphi'$. Our main result is an extension of the above theorem.

**Theorem 2** (I-Yamana). *Let* $\varphi \in \mathscr{A}(G)$ *and* $\varphi' \in \mathscr{A}(G')$. *Then*

$$\mathbf{P}^{G'}_{\text{reg}}(\varphi\otimes\varphi'_s) = I(s,\varphi,\varphi')$$

*as meromorphic functions of* $s$. *Here,* $\mathbf{P}^{G'}_{\text{reg}}$ *is the regularized period integral defined below.*

As immediate consequences, we obtain the following corollaries.

*Corollary* 3.

(1) $\mathbf{P}^{G'}_{\text{reg}}$ is $\Delta G'(\mathbb{A})$-invariant.
(2) $\mathbf{P}^{G'}_{\text{reg}}(\varphi\otimes\varphi'_s)=0$ unless $\varphi$ and $\varphi'$ are generic.

*Corollary* 4. Assume that $\pi$ and $\pi'$ are induced from irreducible cuspidal automorphic representations of Levi subgroups of $G$ and $G'$ respectively. Then $\pi\otimes\pi'$ is $\Delta G'$-distinguished (with respect to $\mathbf{P}^{G'}_{\text{reg}}$) if and only if

$$L\left(\tfrac{1}{2},\pi\times\pi'\right)\neq 0.$$

*Corollary* 5. Let $\varphi\in\pi\subset\mathscr{A}^{\text{disc}}(G)$ and $\varphi'\in\pi'\subset\mathscr{A}^{\text{disc}}(G')$. Here, $\mathscr{A}^{\text{disc}}(G)$ and $\mathscr{A}^{\text{disc}}(G')$ denote the spaces of square integrable automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. Assume that $\pi$ is not 1-dimensional. Then $\mathbf{P}^{G'}(\varphi\otimes\varphi')$ converges and

$$\mathbf{P}^{G'}(\varphi\otimes\varphi') = \mathbf{P}^{G'}_{\text{reg}}(\varphi\otimes\varphi') = 0$$

unless $\pi$ and $\pi'$ are cuspidal.

The original motivation was to study the Gan-Gross-Prasad conjecture in the non-tempered case. We also expect an application to he spectral expansion of the relative trace formula of Jacquet-Rallis [4]. In what follows, we will explain the definition of $\mathbf{P}^{G'}_{\text{reg}}$ and the proof of Theorem 2.

Following Jacquet, Lapid and Rogawski [3], we define $\mathbf{P}^{G'}_{\mathrm{reg}}$. The construction is based on truncation. Recall that Arthur's truncation is given by

$$\Lambda^T\varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{\gamma \in P\backslash G} \varphi_P(\gamma g)\hat{\tau}_P(H_P(\gamma g) - T),$$

which is rapidly decreasing. Here, $P = MU$ is a standard parabolic subgroup of $G$, $\varphi_P$ is the constant term of $\varphi$ along $P$, $\mathfrak{a}_P = \mathrm{Hom}(X^*(M), \mathbb{R})$, $\mathfrak{a}_P^* = X^*(M) \otimes \mathbb{R}$, $\mathfrak{a}_P = \mathfrak{a}_P^G \oplus \mathfrak{a}_G$ is the canonical decomposition, $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ is a function such that $e^{\langle \chi, H_P(m)\rangle} = |\chi(m)|_{\mathbb{A}}$ for $\chi \in X^*(M)$, $m \in M(\mathbb{A})$ and extended by the Iwasawa decomposition, $T \in \mathfrak{a}_0^G = \mathfrak{a}_B^G$ is sufficiently positive with the standard Borel subgroup $B$, and $\hat{\tau}_P$ is the characteristic function of the obtuse cone in $\mathfrak{a}_P$ spanned by coroots. The integral $\mathbf{P}^{G'}(\Lambda^T\varphi \otimes \varphi')$ converges but is hard to compute. Thus we adopt more suitable "mixed truncation" given by

$$\Lambda^T_m\varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{\gamma \in P\backslash PWG'} \varphi_P(\gamma g)\hat{\tau}_P(H_P(\gamma g) - T),$$

where $W$ is the Weyl group of $G$.

**Lemma 6.**

(1) *$\Lambda^T_m\varphi$ is rapidly decreasing on $G'(F)\backslash G'(\mathbb{A})$.*
(2) *$\mathbf{P}^{G'}(\Lambda^T_m\varphi \otimes \varphi') = \sum_\lambda p_\lambda(T)e^{\langle \lambda, T\rangle}$, where the right-hand side is a finite sum with $\lambda \in (\mathfrak{a}^G_{0,\mathbb{C}})^*$ and $p_\lambda \in \mathbb{C}[\mathfrak{a}_0]$.*

We define

$$\mathbf{P}^{G'}_{\mathrm{reg}}(\varphi \otimes \varphi') = p_0(T)$$

if the exponents of $\varphi$ and $\varphi'$ avoid some finitely many hyperplanes. It turns out that $p_0(T)$ is constant, i.e., independent of $T$. If $\varphi \in \mathscr{A}^{\mathrm{cusp}}(G)$, then $\Lambda^T_m\varphi = \varphi$, so that $\mathbf{P}^{G'}_{\mathrm{reg}}(\varphi \otimes \varphi') = \mathbf{P}^{G'}(\varphi \otimes \varphi')$. This identity holds more generally if the exponents of $\varphi$ and $\varphi'$ satisfy some finitely many negativity conditions. We can define $\mathbf{P}^{G'}_{\mathrm{reg}}(\varphi \otimes \varphi'_s)$ for generic $s$ and obtain a meromorphic function of $s$.

Following Lapid and Rogawski [5], we prove Theorem 2. We may assume that $\varphi$ is a cuspidal Eisenstein series. We want to unfold $\mathbf{P}^{G'}(\varphi \otimes \varphi')$ by using the Fourier expansion

$$\varphi(g) = \sum_{i=0}^{n} \sum_{\gamma \in P_i'\backslash G'} W^{\psi}_{Q_i}(\gamma g, \varphi_{Q_i}).$$

Here,

$$Q_i = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{matrix} i \\ n+1-i \end{matrix} \right\} \subset G,$$

$$P'_i = \left\{ \begin{pmatrix} * & * \\ 0 & \nabla \end{pmatrix} \begin{matrix} i \\ n-i \end{matrix} \,\middle|\, \nabla \text{ is upper triangular unipotent} \right\} \subset G',$$

and $W^{\psi}_{Q_i}$ is the Whittaker function for the $\mathrm{GL}_{n+1-i}$ part. If $\varphi \in \mathscr{A}^{\mathrm{cusp}}(G)$, then only the term $i = 0$ survives. Since $P'_0 = N'$ and $W^{\psi}_{Q_0} = W^{\psi}$, we can unfold $\mathbf{P}^{G'}(\varphi \otimes \phi')$ to get $I(s, \varphi, \varphi')$. In general, we cannot unfold. Instead, we compute the convergent integral $\mathbf{P}^{G'}(\theta_\phi \otimes \varphi')$ in two ways. Here, $\phi(\lambda) = f(\lambda) \cdot \varphi$ for $\lambda \in (\mathfrak{a}^G_{P,\mathbb{C}})^*$ with $f \in \mathcal{PW}((\mathfrak{a}^G_{P,\mathbb{C}})^*)$ and $\varphi \in \mathscr{A}^{\mathrm{cusp}}_P(G)$, $\theta_\phi$ is a pseudo Eisenstein series given by

$$\theta_\phi(g) = \int_{\Re\lambda=\kappa} f(\lambda) E(g, \varphi, \lambda)\, \mathrm{d}\lambda$$

with sufficiently positive $\kappa \in (\mathfrak{a}^G_P)^*$ and an Eisenstein series

$$E(g, \varphi, \lambda) = \sum_{\gamma \in P \backslash G} \varphi(\gamma g) e^{\langle \lambda, H_P(\gamma g) \rangle}.$$

We can show that

$$\mathbf{P}^{G'}(\theta_\phi \otimes \varphi'_s) = \int_{\Re\lambda=\kappa} f(\lambda) \mathbf{P}^{G'}_{\mathrm{reg}}(E(\varphi, \lambda) \otimes \varphi'_s)\, \mathrm{d}\lambda$$

under some mild condition of $f$. We can unfold $\mathbf{P}^{G'}(\theta_\phi \otimes \varphi'_s)$ to get

$$\int_{\Re\lambda=\kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi')\, \mathrm{d}\lambda + \sum_{i=1}^{n} \cdots,$$

where the last sum vanishes under another mild condition of $f$. The upshot is

$$\int_{\Re\lambda=\kappa} f(\lambda) \mathbf{P}^{G'}_{\mathrm{reg}}(E(\varphi, \lambda) \otimes \varphi'_s)\, \mathrm{d}\lambda = \int_{\Re\lambda=\kappa} f(\lambda) I(s, E(\varphi, \lambda), \varphi')\, \mathrm{d}\lambda$$

for sufficiently many $f$ which allows us to extract the desired identity.

## References


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